Mixing Processes

For two σ-fields $\mathcal{F}$ and $\mathcal{G}$ we define

$$\alpha(\mathcal{F},\mathcal{G}) = \sup_{F\in\mathcal{F},\, G\in\mathcal{G}} |P(F\cap G) - P(G)P(F)|$$

And define for $r\le s$ $\quad B_r^s = \sigma(X_r, \dots, X_s)$.

Def. - (Strong mixing coefficient) $\quad \alpha(m) = \sup_j \alpha(B_{-\infty}^{j}, B_{j+m}^{\infty})$

Def. - (Strong mixing process) A process $\{X_t\}$ is strong or α-mixing if

$$\lim_{m\to\infty} \alpha(m) = 0.$$

Def. - (Mixing size) $\alpha(m)$ is of size $-a$, where $a>0$ if

$$\alpha(m) = O(m^{-(a+\varepsilon)}) \quad \text{for some } \varepsilon>0$$
$$= K\, m^{-a-\varepsilon}.$$

Proposition 1. - Let $\{X_t\}$ be α-mixing of size $-a$, and $Y_t = g(X_t, \dots, X_{t-h})$ where $g(\cdot)$ is measurable. Then $\{Y_t\}$ is also α-mixing of size $-a$.

proof: For $r\le s$ let $B_r^s = \sigma(X_r,\dots,X_s)$ and $C_r^s = \sigma(Y_r,\dots,Y_s)$. It can be inferred that $C_{-\infty}^{j} \subset B_{-\infty}^{j}$ and $C_{j+m}^{\infty} \subset B_{j+m-h}^{\infty}$ for all j and $m\ge h$.

$$\alpha_Y(m) = \sup_j \sup_{F\in C_{-\infty}^{j},\, G\in C_{j+m}^{\infty}} |P(F\cap G) - P(G)P(F)|$$

$$\le \sup_j \sup_{F\in B_{-\infty}^{j},\, G\in B_{j+m}^{\infty}} |P(F\cap G) - P(G)P(F)|$$

$$= \alpha_X(m-h)$$

$$(m-h)^2 = m^2 - 2mh + h^2 = O(m^2)$$

$$= O(m^{-a-\varepsilon}) \quad \text{for } \varepsilon>0 \text{ and } m>h.$$

Covariance Inequalities for Mixing Processes

Proposition 2.- Suppose that $|X_t| \leq c_1$ and $|X_{t-h}| \leq c_2$ for some $c_1, c_2 > 0$. Then

$$|Cov(X_t, X_{t-h})| \leq 4\, c_1 c_2\, \alpha(h).$$

proof:

Define the following random variable

$$\eta = \text{sign}\{E(X_t | B_{-\infty}^{t-h}) - EX_t\}, \quad \eta \in \{-1, 0, 1\} \text{ and it's } B_{-\infty}^{t-h} \text{ measurable.}$$

(*) Notice that we can write covariances by only demeaning one of the two random variables

$$\begin{aligned} E(X_t - EX_t)(X_{t-h} - EX_{t-h}) &= E\{X_t . X_{t-h} - X_t . EX_{t-h} - EX_t . X_{t-h} + EX_t . EX_{t-h}\} \\ &= E\; X_t . X_{t-h} - EX_t . EX_{t-h} \\ &= E\{X_t (X_{t-h} - EX_{t-h})\} = E\{X_{t-h}(X_t - EX_t)\}. \end{aligned}$$

Now, write

$$\begin{aligned} |Cov(X_t, X_{t-h})| = |E\{X_{t-h}(X_t - EX_t)\}| &\underset{LIE}{=} |E\{X_{t-h}(E(X_t|B_{-\infty}^{t-h}) - EX_t)\}| \\ &\leq E\{|X_{t-h}| . |E(X_t|B_{-\infty}^{t-h}) - EX_t|\} \\ &\leq c_2\; E\{\eta\, (E(X_t|B_{-\infty}^{t-h}) - EX_t)\} \\ &\underset{LIE}{=} c_2\,(E\eta X_t - E\eta EX_t) \\ &= c_2\; Cov(\eta, X_t) \\ &= c_2\; |Cov(\eta, X_t)| \end{aligned}$$

it's non negative by construction!

Next, define the following random variable

$$\xi = \text{sign}\{E(\eta | B_t^{\infty}) - E\eta\}, \quad \xi \in \{-1, 0, 1\} \text{ and it's } B_t^{\infty} \text{ measurable.}$$

Now, repeating the same argument

$$\begin{aligned} |Cov(\eta, X_t)| = |E X_t(\eta - E\eta)| &\underset{LIE}{=} |E\{X_t\,(E(\eta|B_t^{\infty}) - E\eta)\}| \\ &\leq E\{|X_t|\; |E(\eta|B_t^{\infty}) - E\eta|\} \\ &\leq c_1\; E\{\xi\,(E(\eta|B_t^{\infty}) - E\eta)\} \\ &= c_1\,(E\xi\eta - E\xi E\eta) \\ &= c_1\; |E\eta\xi - E\xi E\eta|. \end{aligned}$$

Combining both results yield

$$|\mathrm{Cov}(X_t, X_{t-h})| \leq C_1 \cdot C_2 \, |E\xi\eta - E\xi E\eta| .$$

Define the following events $A_1 = \{\eta = 1\}$ $A_{-1} = \{\eta = -1\}$ $A_0 = \{\eta = 0\}$

$B_1 = \{\xi = 1\}$ $B_{-1} = \{\xi = -1\}$ $B_0 = \{\xi = 0\}$

- $$\begin{aligned} E\eta\xi = \; & P(A_1 \cap B_1) \cdot 1\times 1 + P(A_1 \cap B_{-1})\, 1\times(-1) + \cancel{P(A_1 \cap B_0)}\, 1\times 0 + \\ & P(A_{-1} \cap B_1) \cdot 1\times(-1) + P(A_{-1} \cap B_{-1})\, (-1)\times(-1) + \cancel{P(A_{-1} \cap B_0)}\, (-1)\times 0 + \\ & \cancel{P(A_0 \cap B_1)} \cdot 0\times 1 + \cancel{P(A_0 \cap B_{-1})}\, 0\times(-1) + \cancel{P(A_0 \cap B_{-1})}\, 0\times(-1) \end{aligned}$$

$$= P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1})$$

- $E\eta = P(A_1) - P(A_{-1})$

- $E\xi = P(B_1) - P(B_{-1})$

Hence,

$$\begin{aligned} |E\eta\xi - E\eta E\xi| &= | P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) \\ & \quad - [P(A_1) - P(A_{-1})] \times [P(B_1) - P(B_{-1})] | \\ &= | P(A_1 \cap B_1) + P(A_1 \cap B_{-1}) - P(A_1 \cap B_1) - P(A_1 \cap B_{-1}) \\ & \quad - P(A_1)P(B_1) + P(A_1)P(B_{-1}) + P(A_{-1})P(B_1) - P(A_{-1})P(B_{-1}) | \\ &\leq 4 \sup_{A \in \mathcal{B}_{-\infty}^{t-h},\, B \in \mathcal{B}_t^{\infty}} |P(A \cap B) - P(A)P(B)| \\ &= 4\,\alpha(h). \end{aligned}$$

■

Proposition 3.- Suppose that $E|X_{t-h}|^p < \Delta$ for some $p > 1$ and $|X_t| < C$. Then

$$|\mathrm{Cov}(X_t, X_{t-h})| \leq 6\, C\, \Delta^{1/p} \alpha(h)^{1-1/p}$$

proof: Define

$$B = \left(\frac{E|X_{t-h}|^p}{\alpha(h)} \right)^{1/p}$$

$$X_{t-h}^B = X_{t-h} \, \mathbb{1}\{|X_{t-h}| \leq B\}$$ → Truncation, so now this is a bounded r.v.

$$\tilde{X}_{t-h}^B = X_{t-h} - X_{t-h}^B = X_{t-h}\, \mathbb{1}\{|X_{t-h}| > B\}$$ → Tail part

Then, write

$$\begin{aligned} |\mathrm{Cov}(X_t, X_{t-h})| &\leq |\mathrm{Cov}(X_t, X_{t-h}^B)| + |\mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)| \\ &\underset{\text{using Proposition 2}}{\leq} 4\, C\, B\, \alpha(h) + |\mathrm{Cov}(X_t, \tilde{X}_{t-h}^B)| \end{aligned}$$

$$= 4C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p} + \underbrace{|Cov(X_t, \tilde{X}_{t-h}^B)|}_{\text{we need to bound this new term here.}}$$

$$|Cov(X_t, \tilde{X}_{t-h}^B)| = |E\; X_t\,(\tilde{X}_{t-h}^B - E\tilde{X}_{t-h}^B)|$$

$$\leq E\{|X_t|\,.\,|\tilde{X}_{t-h}^B - E\tilde{X}_{t-h}^B|\}$$

$$\leq C\; E|\tilde{X}_{t-h}^B - E\,\tilde{X}_{t-h}^B| \leq C[E\,|\tilde{X}_{t-h}^B| + |E\,\tilde{X}_{t-h}^B|]$$

$$\leq 2C\,E|\tilde{X}_{t-h}^B|$$

$$= 2C\,E\,[\,|X_{t-h}|\,\mathbb{1}\{|X_{t-h}|>B\}\,]$$

exponent doesn't affect indicator

$$\leq 2C\,(E|X_{t-h}|^p)^{1/p}\,(E\,\mathbb{1}\{|X_{t-h}|>B\}^{p/p-1})^{1-1/p} \qquad \frac{p-1}{p} = 1/q$$

Hölder's Inequality

$$1/p + 1/q = 1,\; p>1$$

$$= 2C\,(E|X_{t-h}|^p)^{1/p}\,\underbrace{P(|X_{t-h}|>B)}^{1-1/p}$$

$$\leq 2C\,(E|X_{t-h}|^p)^{1/p}\left(\frac{E|X_{t-h}|^p}{B^p}\right)^{1-1/p = \frac{p-1}{p}}$$

Markov Inequality

$$= 2C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p}$$

replace B

Combining both results yield

$$|Cov(X_t, X_{t-h})| \leq 6C\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p}.$$

<u>Proposition 4</u>.- Suppose $E|X_{t-h}|^p < \Delta$ and $E|X_t|^p < \Delta$ uniformly over t and also $p>2$ for some $\Delta>0$. Then,

$$|Cov(X_t, X_{t-h})| \leq 8\,\Delta^{2/p}\,\alpha(h)^{1-2/p}.$$

<u>proof:</u> Again define $B = \left(\frac{E|X_{t-h}|^p}{\alpha(h)}\right)^{1/p}$.

$$|Cov(X_t, X_{t-h})| = |Cov(X_t,\; X_{t-h}^B + \tilde{X}_{t-h}^B)|$$

truncation trick

$$= |Cov(X_t, X_{t-h}^B) + Cov(X_t, \tilde{X}_{t-h}^B)|$$

p moment, bounded

$$\leq |Cov(X_t, X_{t-h}^B)| + |Cov(X_t, \tilde{X}_{t-h}^B)|$$

$$\leq 6\,B\,(E|X_{t-h}|^p)^{1/p}\,\alpha(h)^{1-1/p} + |Cov(X_t, \tilde{X}_{t-h}^B)|$$

Proposition 3

$$\le \quad 6\,\Delta^{2/p}\,\alpha(h)^{1-2/p} + \underbrace{|\mathrm{Cov}(X_t, \tilde{X}_{t-h}^{B})|}_{\text{We need to bound this term here.}}$$

replace B and use tail condition on $|X_{t-h}|$

Notice that $|X_t|$ is not bounded like in Proposition 3.

$$|\mathrm{Cov}(X_t, \tilde{X}_{t-h}^{B})| = |E(\tilde{X}_{t-h}^{B}(X_t - EX_t))|$$

$$\le \left(E|X_t - EX_t|^p\right)^{1/p} \left(E|\tilde{X}_{t-h}^{B}|^{\frac{p}{p-1}}\right)^{1-1/p \to \frac{p-1}{p}}$$

Hölder's inequality

$$\le \left[(E|X_t|^p)^{1/p} + (|EX_t|^p)^{1/p}\right] \left(E|\tilde{X}_{t-h}^{B}|^{\frac{p}{p-1}}\right)^{1-1/p}$$

Minkowski's Inequality (fancy triangle ineq) $\|X+Y\|_p \le \|X\|_p + \|Y\|_p$

$$\le 2\,(E|X_t|^p)^{1/p} \left(E|\tilde{X}_{t-h}^{B}|^{\frac{p}{p-1}}\right)^{1-1/p}$$

$$\le 2\,\Delta^{1/p} \left(E(|X_{t-h}|^{\frac{p}{p-1}}\, \mathbb{1}\{|X_{t-h}|>B\})\right)^{1-1/p = \frac{p-1}{p}}$$

$$\le 2\,\Delta^{1/p} \left\{ \left(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}}\right)^{1/\bar{p}} \left(P(|X_{t-h}|>B)\right)^{1-1/\bar{p} = \frac{\bar{p}-1}{\bar{p}}} \right\}^{1-1/p}$$

Hölder's inequality

$$\le 2\,\Delta^{1/p} \left\{\left(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}}\right)^{\frac{p-1}{p}\cdot\frac{1}{\bar{p}}}\right\} \left\{\frac{E|X_{t-h}|^p}{B^p}\right\}^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}$$

Markov's Inequality

$$= 2\,\Delta^{1/p} \left\{\left(E|X_{t-h}|^{\frac{p}{p-1}\cdot\bar{p}}\right)^{\frac{p-1}{p}\cdot\frac{1}{\bar{p}}}\right\} \underbrace{\alpha(h)^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}} \left\{\frac{E|X_{t-h}|^p}{(E|X_{t-h}|^p)}\right\}^{\frac{\bar{p}-1}{\bar{p}}\cdot\frac{p-1}{p}}$$

we want this to be $1-\frac{2}{p} = \frac{p-2}{p}$

then $\frac{\bar{p}-1}{\bar{p}} \cdot \frac{p-1}{p} = \frac{p-2}{p} \Rightarrow \bar{p} = p-1$

To use Hölder we need $\bar{p} > 1 \Leftrightarrow p > 2$.

$$= 2\,\Delta^{1/p} \left(E|X_{t-h}|^p\right)^{1/p} \alpha(h)^{1-\frac{2}{p}}$$

Setting $\bar{p} = p-1$

$$\le 2\,\Delta^{1/p}\,\Delta^{1/p}\,\alpha(h)^{1-2/p}$$

$$\le 2\,\Delta^{2/p}\,\alpha(h)^{1-2/p}.$$

Combining the two results leads us to

$$|\mathrm{Cov}(X_t, X_{t-h})| \le 8\,\Delta^{2/p}\,\alpha(h)^{1-\frac{2}{p}}. \quad \blacksquare$$

Corollary 1 :- Suppose $\{X_t\}$ is α-mixing and for some $p > 2$:

(i) $\alpha(h)$ is of size $\frac{-p}{p-2}$

(ii) $E|X_t|^p < \Delta$ for all t $\Leftrightarrow$ $\sup_t E|X_t|^p < \Delta < \infty$

Then
$$\Omega_n := \mathrm{Var}\left(\frac{1}{\sqrt{n}} \sum X_t\right) = O(1).$$

proof:

$$\Omega_n = \frac{1}{n}\sum_{t=1}^{n} \mathrm{Var}(X_t) + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} \mathrm{Cov}(X_t, X_{t-h})$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} \mathrm{Var}(X_t) + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} |\mathrm{Cov}(X_t, X_{t-h})|$$

$$= \frac{1}{n}\sum_{t=1}^{n} \left[EX_t^2 - (EX_t)^2\right] + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} |\mathrm{Cov}(X_t, X_{t-h})|$$

$$\leq \frac{1}{n}\sum_{t=1}^{n} \left[(EX_t^2)^{1/2}\right]^2 + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} |\mathrm{Cov}(X_t, X_{t-h})|$$

$\|X\|_p \leq \|X\|_q$ for $q \geq p$

norm inequality
$$\leq \frac{1}{n}\sum_{t=1}^{n} (E|X_t|^p)^{2/p} + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} |\mathrm{Cov}(X_t, X_{t-h})|$$

$$\leq \Delta^{2/p} + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} |\mathrm{Cov}(X_t, X_{t-h})|$$

⊛ $\alpha(h)$ size $-a$ means $\alpha(h) = O(m^{-a-\varepsilon})$

Proposition 4
$$\leq \Delta^{2/p} + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} 8\,\Delta^{2/p}\, \alpha(h)^{1-\frac{2}{p}}$$

$\delta = a + \varepsilon$ where $-a$ is the size of mixing coefficients

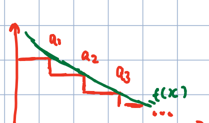
property of α coefficient
$$= \Delta^{2/p} + 2\sum_{h=1}^{n-1} \frac{1}{n} \sum_{t=h+1}^{n} \underbrace{8\,\Delta^{2/p}\, K}_{K}\, h^{-(1-2/p)\underbrace{\delta}_{(a+\varepsilon)}} \quad \text{for some } K > 0.$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1} \frac{1}{n} \left(n - (h+\cancel{1}) + \cancel{1}\right) h^{-\delta(1-2/p)}$$

$$\leq \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1} \frac{1}{\cancel{n}} \cdot \cancel{n}\; h^{-\underbrace{\delta(1-2/p)}_{\gamma}}$$

$$= \Delta^{2/p} + 16\,\Delta^{2/p} K \sum_{h=1}^{n-1} h^{-\gamma} \quad \text{where } \gamma = \delta(1-2/p).$$

⊛ For a decreasing sequence $a_n \geq a_{n+1} \geq \dots \geq 0$ we have $\sum_{n=2}^{n} a_n \leq \int_1^n f(x)\,dx$



s.t. $f(\cdot)$ is non increasing, continous and $f(h) = a_h$.

$$= \Delta^{2/p} + 16\, \Delta^{2/p} K \left[1 + \sum_{h=2}^{n-1} h^{-\gamma} \right]$$

$$\underset{\text{using integral approximation}}{\leq} \Delta^{2/p} + 16\, \Delta^{2/p} K \left[1 + \int_1^n x^{-\gamma}\, dx \right]$$

$$= \Delta^{2/p} + 16\, \Delta^{2/p} K \left[1 + \frac{x^{-\gamma+1}}{1-\gamma} \Big|_1^n \right]$$

$$\underset{\text{provided } \gamma \neq 1}{=} \Delta^{2/p} + \underbrace{16\, \Delta^{2/p} K}_{\text{some constant}} \frac{n^{-\gamma+1}}{1-\gamma}$$

$$= O(n^{1-\gamma})$$ if I divide by $n^{1-\gamma}$ we get some constant

since $\gamma \neq 1$ in order to not diverge we must have $\gamma > 1$, which requires

$$\underbrace{\delta}_{(a+\varepsilon)} (1 - 2/p) > 1$$

where a is the size and $\varepsilon > 0$.

$$\underbrace{a\left(\frac{p-2}{p}\right)}_{\substack{\text{the smallest number} \\ \text{possible for this is } 1}} + \underbrace{\varepsilon\left(\frac{p-2}{p}\right)}_{\text{some positive number}} > 1$$

$= 1$

: ε is some > 0

$$\Rightarrow \quad a = \frac{p}{p-2}$$

And since the size of the α-mixing coefficients is $-a$ the size that we need is $\frac{-p}{p-2}$. ■

⊛ What happens when $\gamma = 1$? Then $\frac{n^{1-\gamma}}{1-\gamma} \to \ln(n)$ as $\gamma \to 1$ which is divergent.

Proposition 5.- Suppose that $\{X_t : t=1,\dots,n\}$ is an α-mixing sequence of random variables such that for some $\Delta>0$, $\delta>0$ and $\eta>0$,

$$\underbrace{\alpha(h) \le \Delta\, h^{-\delta}}_{=O(h^{-\delta})} \quad (\text{any size of } \alpha\text{-mixing})$$

$$E|X_t|^{1+\eta} \le \Delta \quad \text{for all } t \iff \sup_t E|X_t|^{1+\eta} \le \Delta.$$

Then,

$$\frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \xrightarrow{p} 0$$

proof:

$$P\left(\left| \frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \right| > \varepsilon \right) \underset{\text{Markov's Inequality}}{\le} \frac{1}{\varepsilon} \underbrace{\overbrace{E\left| \frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \right|}^{<\varepsilon^2}}_{\text{We need this to go to zero as } n\to\infty.} < \varepsilon$$

Since we are not assuming second or higher moments we need to use the truncation trick.
Define

$$X_t^B = X_t\, \mathbb{1}\{|X_t| \le B\}$$

$$\tilde{X}_t^B = X_t\, \mathbb{1}\{|X_t| > B\}$$

Then we write

We want the sum to be $<\varepsilon^2$

$$E\left| \frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \right| \le \underbrace{E\left| \frac{1}{n}\sum_{t=1}^{n} X_t^B - \frac{1}{n}\sum_{t=1}^{n} EX_t^B \right|}_{A \;<\varepsilon^2/2} + \underbrace{E\left| \frac{1}{n}\sum_{t=1}^{n} \tilde{X}_t^B - \frac{1}{n}\sum_{t=1}^{n} E\tilde{X}_t^B \right|}_{B \;<\varepsilon^2/2}$$

We will work with B first, to know when we must truncate

$$\begin{aligned}
E\left| \frac{1}{n}\sum_{t=1}^{n} \tilde{X}_t^B - \frac{1}{n}\sum_{t=1}^{n} E\tilde{X}_t^B \right| &\le 2\,\frac{1}{n}\sum_{t=1}^{n} E|\tilde{X}_t^B| \\
&\le 2 \sup_t E|\tilde{X}_t^B| \\
&= 2 \sup_t E\left[|X_t|\, \mathbb{1}\{|X_t|>B\} \right] \\
&\underset{\text{Hölder's Inequality}}{\le} 2 \sup_t \left(E|X_t|^{1+\eta} \right)^{\frac{1}{1+\eta}} \left(P(|X_t|>B) \right)^{\frac{\eta}{1+\eta}} \qquad \frac{1}{1+\eta},\ \frac{\eta}{1+\eta} = \frac{1+\eta-1}{1+\eta} \\
&\underset{\text{Markov's Inequality}}{\le} 2 \sup_t \left(E|X_t|^{1+\eta} \right)^{\frac{1}{1+\eta}} \left(\frac{E|X_t|^{1+\eta}}{B^\eta} \right)^{\frac{\eta}{1+\eta}} \\
&= 2 \sup_t \left(E|X_t|^{1+\eta} \right) \cdot \frac{1}{B^\eta} \qquad \text{I'm looking for a } B \text{ to choose.} \\
&\le \frac{2\Delta}{B^\eta}
\end{aligned}$$

We need it to be less than $\varepsilon^2/2$ which requires $\frac{2\Delta}{B^\eta} < \varepsilon^2/2 \Rightarrow B > \left(\frac{4\Delta}{\varepsilon^2} \right)^{1/\eta}$

now we will work with A that involve random variables bounded by the B we just defined. This means that their moments are finite. We could make use of that.

$$\left(E\left|\frac{1}{n}\sum_{t=1}^{n}(X_t^B - EX_t^B)\right|\right)^{1/1} \underset{\text{norm inequality}}{\leq} \left\{E\left|\frac{1}{n}\sum_{t=1}^{n}(X_t^B - EX_t^B)\right|^2\right\}^{1/2} \qquad \|X\|_1 \leq \|X\|_2$$

$$= \left\{\frac{1}{n}\underbrace{Var\left(\frac{1}{\sqrt{n}}\sum X_t^B\right)}_{C}\right\}^{1/2}$$

Remember that

$$Var\left(\frac{1}{\sqrt{n}}\sum X_t^B\right) = \frac{1}{n}\sum_{t=1}^{n} Var(X_t^B) + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} Cov(X_t^B, X_{t-h}^B)$$

$$\leq \underbrace{\frac{1}{n}\sum_{t=1}^{n} E|X_t^B|^2}_{\frac{1}{n} n B^2 = B^2} + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n}\left|Cov(X_t^B, X_{t-h}^B)\right|$$

$$\underset{\text{Proposition 2}}{\leq} B^2 + 2\sum_{h=1}^{n-1}\frac{1}{n}\sum_{t=h+1}^{n} 4B^2\alpha(h)$$

$$= B^2 + 8B^2\sum_{h=1}^{n-1}\frac{1}{n}(n-(h+1)+1)\alpha(h)$$

$$\leq B^2 + 8B^2\sum_{h=1}^{n-1}\alpha(h)$$

$$\leq B^2 + 8B^2\Delta\sum_{h=1}^{n-1}h^{-\delta}$$

$$= B^2 + 8B^2\Delta\left[1 + \sum_{h=2}^{n-1}h^{-\delta}\right]$$

$$\underset{\text{integral approximation}}{\leq} B^2 + 8B^2\Delta\left[1 + \int_1^n x^{-\delta}dx\right]$$

$$\underset{\text{provided } \delta\neq 1}{=} B^2 + 8B^2\Delta\left[1 + \frac{x^{-\delta+1}}{1-\delta}\Big|_1^n\right]$$

$$= O(n^{1-\delta})$$

Using this result yields

$$E\left|\frac{1}{n}\sum_{t=1}^{n}(X_t^B - EX_t^B)\right| \leq \left\{\frac{1}{n}O(n^{1-\delta})\right\}^{1/2}$$

$$= \underbrace{O(n^{-\delta/2})}_{\to 0}$$

For this to go to zero we require $\delta = a + \varepsilon > 0$, $\varepsilon > 0$
so for any size c of the mixing coefficients this condition will hold!

Putting it all together gives

$$\lim_{n\to\infty} P\left(\left| \frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \right| > \varepsilon \right) \leq \frac{1}{\varepsilon} E \left| \frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \right|$$

$$\leq \frac{1}{\varepsilon} E\left| \frac{1}{n}\sum_{t=1}^{n} X_t^B - \frac{1}{n}\sum_{t=1}^{n} EX_t^B \right| + E\left| \frac{1}{n}\sum_{t=1}^{n} \tilde{X}_t^B - \frac{1}{n}\sum_{t=1}^{n} E\tilde{X}_t^B \right| \cdot \frac{1}{\varepsilon}$$

$$\underset{\lim_{n\to\infty}}{\leq} \frac{1}{\varepsilon} \cdot o(1) + \frac{1}{\varepsilon} \varepsilon^2/2 < \varepsilon/2 < \varepsilon \, . \, \square$$

Then by taking limits we get the desired result

$$\lim_{n\to\infty} P\left(\left| \frac{1}{n}\sum_{t=1}^{n} X_t - \frac{1}{n}\sum_{t=1}^{n} EX_t \right| > \varepsilon \right) \leq \varepsilon/2 \, . \quad \blacksquare$$

<u>Lemma 1</u>.- (CLT) Let $\{X_{nt}\}$ be a sequence such that $EX_{nt} = 0$ for all n, t and

(i) α coefficients are of size $\frac{-p}{p-2}$, $p > 2$

(ii) $\sup_t E|X_{nt}|^p < \Delta$ for all n

(iii) $w_n := \mathrm{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \right) > \delta > 0$ for all n sufficiently large

Then

$$\frac{1}{\sqrt{n}} \sum_{t=1}^{n} \frac{X_{nt}}{w_n^{1/2}} \xrightarrow{d} N(0,1) \, . \longrightarrow \text{CLT for r.v.}$$

CLT for r. vectors we have to make them r.v.

<u>Definition</u>.- Let $\{M_n\}$ be a sequence of $k \times k$ matrices. Let $\underline{e}_n$ be the smallest eigenvalue of M_n. Then M_n is said to be uniformly positive definite if for all n sufficiently large $\underline{e}_n > \delta > 0$ uniformly in n.

<u>Proposition 6</u>.- Let $\{X_{nt}\}$ be an α-mixing sequence of random vectors such that $EX_{nt} = 0$ for all n,t and for some $p > 2$ and $\Delta > 0$,

(*) Recall $X_{nt} = \begin{pmatrix} X_{nt1} \\ \vdots \\ X_{ntK} \end{pmatrix}_{K \times 1}$

(i) α is of size $\frac{-p}{p-2}$

(ii) $E|X_{nt}|^p \leq \Delta$ for all t, n

(iii) $\Omega_n = \mathrm{Var}\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \right)$ is uniformly positive definite.

Then

$$\Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \xrightarrow{d} N(0, I_K)$$

Proof: Let $\lambda \in \mathbb{R}^k$ such that $\|\lambda\| = 1$. Then by the Cramér-Wold device we want to show that $\lambda' \Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt} \xrightarrow{d} N(0,1)$ using Lemma 1.

$= \frac{1}{\sqrt{n}} \sum \lambda' \Omega_n^{-1/2} X_{nt}$

We need to check for the conditions in Lemma 1

- $E(\lambda' \Omega_n^{-1/2} X_{nt}) = \lambda' \Omega_n^{-1/2} E(X_{nt}) \Omega_n^{-1/2} \lambda = 0$ for all n,t.

$\underset{k\times 1}{\vec{X}_{nt}} = \begin{pmatrix} X_{nt1} \\ \vdots \\ X_{ntk} \end{pmatrix}$

(i) • Define the process $\{Y_{nt}\} := \{\lambda' \Omega_n^{-1/2} X_{nt}\} = g(X_{nt1}, \dots, X_{ntk})$ where g is measurable. Then by Proposition 1 $\{Y_{nt}\}$ is α-mixing of the same size $\frac{-p}{p-2}$.

(ii) •
$$\sup_t E|\underbrace{\lambda' \Omega_n^{-1/2}}_{\substack{1\times k \\ \text{that depends} \\ \text{on } n}} \underset{k\times 1}{X_{nt}}|^p = \sup_t E\left|\sum_{j=1}^{k} c_{jn} X_{ntj}\right|^p$$

$$= \sup_t \left(E\left|\sum_{j=1}^{k} c_{jn} X_{ntj}\right|^p\right)^{1/p \cdot p}$$

$$\underset{\text{Minkowski's Inequality}}{\leq} \sup_t \left\{\sum_{j=1}^{k} |c_{jn}| \left(E|X_{ntj}|^p\right)^{1/p}\right\}^p$$

$$\leq \Delta \left(\sum_{j=1}^{k} \underbrace{|c_{jn}|}_{L_1 \text{ norm}}\right)^p$$

$$\underset{\text{Norm inequality}}{\leq} \Delta \left(\sum_{j=1}^{k} |c_{jn}|^2\right)^{p/2}$$

$$= \Delta (\lambda' \Omega_n^{-1} \lambda)^{p/2}$$

$C_n \Lambda_n^{-1} C_n'$ where $C_n C_n' = I_k$, $\|\lambda\| = 1$

$$\underset{\text{spectral decomposition}}{=} \Delta (\underbrace{\lambda' C_n}_{d_n'} \Lambda_n^{-1} \underbrace{C_n' \lambda}_{d_n})^{p/2}$$

where $d_n' d_n = 1$ by construction

$$= \Delta (d_n' \Lambda_n^{-1} d_n)^{p/2}$$

largest eigenv of Ω_n^{-1} is the smallest eigenv of Ω

$$\leq \Delta \Big(\underset{<\delta}{e_n^{-1}} \sum_{t=1}^{n} d_{nt}^2\Big)^{p/2}$$

e_n^{-1} is the largest eigenvalue of Ω_n^{-1}

$\|d_n\| = d_n' d_n = 1$

$$< \Delta \, \delta^{-p/2}$$

$$< \infty.$$

- $Var\left(\lambda' \Omega_n^{-1/2} \frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt}\right) = \lambda' \Omega_n^{-1/2} Var\left(\frac{1}{\sqrt{n}} \sum_{t=1}^{n} X_{nt}\right) \Omega_n^{-1/2} \lambda$

$$= \lambda' \Omega_n^{-1/2} \Omega_n \Omega_n^{-1/2} \lambda = 1 > 0$$ no matter n, t

Therefore, by Lemma 1 the desired result holds and the proof is complete. ∎