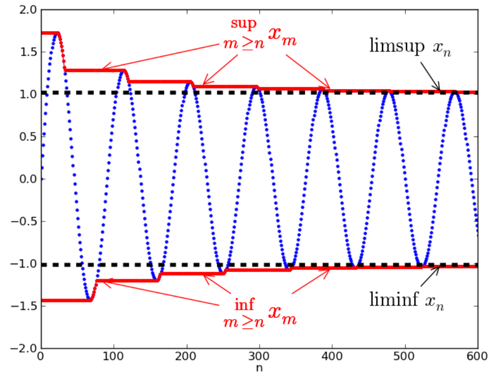
Let's begin with some definitions. Let x_n and a_n be a sequence of constants.

- $x_n = o(a_n)$ means $\frac{x_n}{a_n} \to 0$ as $n \to \infty$.

- $x_n = O(a_n)$ means $\left\| \frac{x_n}{a_n} \right\| \leq M$

- $\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup_{m \geq n} x_m = \inf_n \sup_{m\geq n} x_m \equiv \overline{\lim_{n\to\infty}} x_n$

  $\liminf_{n\to\infty} x_n = \lim_{n\to\infty} \inf_{m\geq n} x_m = \sup_n \inf_{m \geq n} x_m \equiv \underline{\lim_{n\to\infty}} x_n$



(Taken from Wikipedia)

Now let x_n be a sequence of random vectors.

- $x_n = o_p(a_n)$ means $\frac{x_n}{a_n} \xrightarrow{p} 0 \equiv \forall \varepsilon \lim_{n\to\infty} P\left\{ \left\| \frac{X_n}{a_n} \right\| > \varepsilon \right\} = 0$

- $x_n = O_p(a_n)$ means $\forall \varepsilon, \exists M_\varepsilon > 0$ such that $\limsup_{n\to\infty} P\left\{ \left\| \frac{x_n}{a_n} \right\| > M_\varepsilon \right\} < \varepsilon$

- Continuity at θ: $\forall \varepsilon, \exists \delta$ s.t. $\sup_{\theta' \in B(\theta,\delta)} | H(\theta) - H(\theta') | < \varepsilon$

- Uniform Continuity: $\forall \varepsilon, \exists \delta$ s.t. $\sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta,\delta)} | H(\theta) - H(\theta') | < \varepsilon$

now let $\{H_n(\cdot), n \geq 1\}$ be a family of functions from $\Theta \mapsto \mathbb{R}$.

- (uniform) Equicontinuity : $\forall \varepsilon, \exists \delta$ s.t. $\sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta,\delta)} |h(\theta) - h(\theta')| < \varepsilon \quad \forall h \in \{H_n(\cdot), n \geq 1\}$

now let $\{H_n(\cdot), n \geq 1\}$ is a family of random functions from $\Theta \mapsto \mathbb{R}$.

- Stochastic Equicontinuity : $\forall \varepsilon, \exists \delta$ s.t. $\limsup_{n\to\infty} P\left(\sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta,\delta)} |H_n(\theta) - H_n(\theta')| > \varepsilon \right) < \varepsilon$

If $\{H_n(\cdot), n \geq 1\}$ are vector valued, i.e. $\Theta \mapsto \mathbb{R}^K$, then we use the norm

$$\forall \varepsilon, \exists \delta \text{ s.t. } \limsup_{n\to\infty} P\Big(\underbrace{\sup_{\theta \in \Theta} \sup_{\theta' \in B(\theta,\delta)} \| H_n(\theta) - H_n(\theta') \|}_{\text{Modulus of continuity}} > \varepsilon \Big) < \varepsilon$$

$\otimes \quad V_n(\theta_n) = V_n(\theta_0) + o_p(1)$ if $\theta_n \xrightarrow{p} \theta_0$.

Consistency

(1) EE : $\hat{\theta}_n \in \Theta$: $Q_n(\hat{\theta}_n) \leq \inf_{\theta \in \Theta} Q_n(\theta) + o_p(1)$

(2) UWC : $\sup_{\theta \in \Theta} |Q_n(\theta) - Q(\theta)| \xrightarrow{p} 0$

(3) ID : $\exists\, \theta_0 \in \Theta$ such that $\forall \varepsilon > 0$

$$\inf_{\theta \notin B(\theta_0, \varepsilon)} Q(\theta) > Q(\theta_0)$$

Theorem :- $(1)-(3) \Rightarrow \hat{\theta}_n \xrightarrow{p} \theta_0$.

proof :

$$P(\hat{\theta}_n \notin B(\theta_0,\varepsilon)) \equiv P(\|\hat{\theta}_n - \theta_0\| \geq \varepsilon) \underset{(3)}{\leq} P(Q(\hat{\theta}_n) - Q(\theta_0) \geq \delta) \quad \text{for some } \delta$$

$$= P\left(Q(\hat{\theta}_n) - Q_n(\hat{\theta}_n) + Q_n(\hat{\theta}_n) - Q(\theta_0) \geq \delta \right)$$

Why? $Q_n(\hat{\theta}_n) \leq \inf_{\theta \in \Theta} Q_n(\theta) + o_p(1)$
$\leq Q_n(\theta_0) + o_p(1)$

$$\underset{(1)}{\leq} P\left(Q(\hat{\theta}_n) - Q_n(\hat{\theta}_n) + Q_n(\theta_0) + o_p(1) - Q(\theta_0) \geq \delta \right)$$

$$\leq P\left(|Q(\hat{\theta}_n) - Q_n(\hat{\theta}_n)| + |Q_n(\theta_0) - Q(\theta_0)| + o_p(1) \geq \delta \right)$$

$$\leq P\Big(\underbrace{2 \sup_{\theta \in \Theta} |Q(\theta) - Q_n(\theta)|}_{o_p(1) \text{ by } (2)} + o_p(1) \geq \delta \Big)$$

Then

$$\lim_{n\to\infty} P(\hat{\theta_n} \notin B(\theta_0, \varepsilon)) \leq \lim_{n\to\infty} P(\ o_p(1) \geq \delta) = 0. \quad \blacksquare$$

Supremum/Inf exist always and sup/inf of r.v. are always measurable!

Lemma: - Let $\{X_n\}$ be a sequence of measurable functions then $\sup_n X_n$, $\inf_n X_n$, $\limsup_{n\to\infty} X_n$ and $\liminf_{n\to\infty} X_n$ are measurable functions as well.

proof: Fix $a \in \mathbb{R}$ and let $g = \sup_n X_n$. We claim

$$g^{-1}((a, \infty]) \subseteq \bigcup_{n=1}^{\infty} X_n^{-1}((a, \infty])$$

To see this, notice $\omega \in g^{-1}((a, \infty])$ if and only if

$$g(\omega) > a$$
$$\sup_n X_n(\omega) > a$$

which holds if and only if $X_n(\omega) > a$ for some n. This, at the same time can only happen iff $\omega \in \bigcup_{n=1}^{\infty} X_n^{-1}((a, \infty])$. We can apply the same logic to $-X_n$ to show that infimum is measurable.

To see that $\limsup_{n\to\infty} X_n$ is measurable, write

$$\limsup_{n\to\infty} X_n = \inf_{n\geq 1} \sup_{k\geq n} X_k$$

$$= \inf_{n\geq 1} g_n \text{, where } g_n \text{ is measurable.}$$

Then, by the same logic $\inf_{n\geq 1} g_n$ is also measurable. By the same logic we show that liminf is measurable.

We say a family of random functions $\{ H_n : \Theta \to \mathbb{R}^m, n \geq 1 \}$ is uniformly stochastic equicontinuous if $\forall \varepsilon$, $\exists \delta$ such that

$$\limsup_{n\to\infty} P\Big(\overbrace{\sup_{\theta\in\Theta} \sup_{\theta' \in B(\theta,\delta)} \| H_n(\theta) - H_n(\theta')\|}^{\text{This is called modulus of continuity of } H_n(\cdot)} > \varepsilon \Big) < \varepsilon$$

We will assume $\Theta \subset \mathbb{R}^k$, but it could also be a space of functions $\mathcal{G}$. This can arise, for instance, with efficient instruments $g^*(\cdot)$.

<u>Example 1:</u> $$H_n(\theta) = \frac{1}{n}\sum_{i=1}^n \theta' X_i \qquad \text{(a standard average process)}$$

Let's see if it's SE. We want to show

$$\overline{\lim_{n\to\infty}} P\Big(\sup_{\theta\in\Theta} \sup_{\theta'\in B(\theta,\delta)} |H_n(\theta') - H_n(\theta)| > \varepsilon \Big) < \varepsilon$$

Write

$$H_n(\theta_1) - H_n(\theta_2) = \frac{1}{n}\sum_{i=1}^n (\theta_1 - \theta_2)' X_i$$

$$\Rightarrow |H_n(\theta_1) - H_n(\theta_2)| \leq \frac{1}{n}\sum_{i=1}^n \|\theta_1 - \theta_2\| \, \|X_i\|$$

$$\Rightarrow \sup_{\theta_2\in B(\theta_1,\delta)} |H_n(\theta_1) - H_n(\theta_2)| \leq \frac{1}{n}\sum_{i=1}^n \delta . \|X_i\|$$

$$\Rightarrow \sup_{\theta_1\in\Theta} \sup_{\theta_2\in B(\theta_1,\delta)} |H_n(\theta_1) - H_n(\theta_2)| \leq \underbrace{\frac{1}{n}\sum_{i=1}^n \delta . \|X_i\|}_{= \delta E\|X_i\| + o_p(1)}$$

Therefore,

$$\overline{\lim_{n\to\infty}} P\Big(\sup_{\theta\in\Theta} \sup_{\theta'\in B(\theta,\delta)} |H_n(\theta') - H_n(\theta)| > \varepsilon \Big) \leq \overline{\lim_{n\to\infty}} P\Big(\delta E\|X_i\| + o_p(1) > \varepsilon \Big)$$

$$= \overline{\lim_{n\to\infty}} P\Big(o_p(1) > \underbrace{\frac{\varepsilon}{\delta} - E\|X_i\|}_{\text{choose } \frac{\varepsilon}{\delta} - E\|X_i\| > 0 \Rightarrow \delta < \frac{\varepsilon}{E\|X_i\|}} \Big)$$

$= 0$, so it holds.

Example 2: $H_n(\theta) = \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \theta' (X_i - EX_i)}_{O_p(1) \text{ by CLT at parametric rate } \sqrt{n}.}$ (a centered root n $O_p(1)$ process)

We want to show $\overline{\lim_{n\to\infty}} P\left(\sup_{\theta\in\Theta} \sup_{\theta'\in B(\theta,d)} |H_n(\theta') - H_n(\theta)| > \varepsilon \right) < \varepsilon$.

write

$$H_n(\theta_1) - H_n(\theta_2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\theta_1 - \theta_2)' (X_i - EX_i)$$

$$\Rightarrow |H_n(\theta_1) - M_n(\theta_2)| \leq \|\theta_1 - \theta_2\| \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - EX_i) \right\|$$

$$\Rightarrow \sup_{\theta_2 \in B(\theta_1, d)} |H_n(\theta_1) - M_n(\theta_2)| \leq \delta \underbrace{\left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - EX_i) \right\|}_{O_p(1)}$$

Recall that $Y_n = O_p(1)$ means $\exists M_\varepsilon < \infty$ s.t. $\overline{\lim_{n\to\infty}} P(\|Y_n\| > M_\varepsilon) < \varepsilon$.

write

$$\overline{\lim_{n\to\infty}} P\left(\sup_{\theta_1\in\Theta} \sup_{\theta_2\in B(\theta_1,\delta)} |H_n(\theta_1) - H_n(\theta_2)| > \varepsilon \right) \leq \overline{\lim_{n\to\infty}} P\left(\delta \|O_p(1)\| > \varepsilon \right)$$

choose δ such that $\varepsilon/\delta \geq M_\varepsilon$,

so $\delta \leq \varepsilon / M_\varepsilon$

$= 0$, so it also holds.

Example 3: $\quad H_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \theta' X_i \ , \ E X_i \neq 0$ $\quad$ (non-centered root $\sqrt{n}$ process)

Write

$$H_n(\theta_1) - H_n(\theta_2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\theta_1 - \theta_2)' X_i \pm \sqrt{n}\, (\theta_1 - \theta_2)' E X_i$$

$$= \underbrace{\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\theta_1 - \theta_2)' (X_i - E X_i)}_{O_p(1) \text{ again}} + \sqrt{n}\, (\theta_1 - \theta_2)' E X_i$$

$$\Rightarrow \sup_{\theta_2 \in B(\theta_1, \delta)} | H_n(\theta_1) - H_n(\theta_2) | \leq \delta \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - E X_i) \right\| + \delta \sqrt{n}\, E \|X_i\|$$

$$\Rightarrow \sup_{\theta_1 \in \Theta} \sup_{\theta_2 \in B(\theta_1, \delta)} | H_n(\theta_1) - H_n(\theta_2) | \leq \underbrace{\delta \left\| \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (X_i - E X_i) \right\|}_{\text{Bounded in prob.}} + \underbrace{\delta \sqrt{n}\, E \|X_i\|}_{\text{Explodes}}$$

Then

$$\lim_{n \to \infty} P\left(\sup_{\theta_1 \in \Theta} \sup_{\theta_2 \in B(\theta_1, \delta)} |H_n(\theta_1) - H_n(\theta_2)| > \varepsilon \right) \leq \lim_{n \to \infty} P\left(\delta \|O_p(1)\| + \delta \sqrt{n}\, E\|X_i\| > \varepsilon \right)$$

$= 1$, so we haven't created any useful bound.

Let's bound it the other way:

$$P\left(\sup_{\theta_1 \in \Theta} \sup_{\theta_2 \in B(\theta_1, \delta)} |H_n(\theta_1) - H_n(\theta_2)| > \varepsilon \right) \geq P\left(\sup_{\theta_2 \in B(\theta_1, \delta)} | H_n(\theta_1) - H_n(\theta_2) | > \varepsilon \right)$$

$$= P\left(\sup_{\theta_2 \in B(\theta_1, \delta)} | O_p(1) + \sqrt{n}\, (\theta_1 - \theta_2)' E X_i | > \varepsilon \right)$$

pick any θ_2^* in $B(\theta_1, \delta)$

$$\geq P\left(O_p(1) + \sqrt{n}\, (\theta_1 - \theta_2^*)' E X_i > \varepsilon \right)$$

$$+$$

$$P\left(O_p(1) + \sqrt{n}\, (\theta_1 - \theta_2^*)' E X_i < -\varepsilon \right)$$

$$= P\left(O_p(1) > \varepsilon - \sqrt{n}\,(\theta_1 - \theta_2^*)'\, EX_i \right)$$

$$+$$

$$P\left(O_p(1) < -\varepsilon - \sqrt{n}\,(\theta_1 - \theta_2^*)'\, EX_i \right)$$

CASE 1) $(\theta_1-\theta_2^*)'\, EX_i > 0 \iff -\sqrt{n}\,(\theta_1-\theta_2^*)'\, EX_i \to -\infty$ as $n \to \infty$

then

$$P\left(O_p(1) > \varepsilon - \sqrt{n}\,(\theta_1-\theta_2^*)'\, EX_i \right) \to 1$$

$$P\left(O_p(1) < -\varepsilon - \sqrt{n}\,(\theta_1-\theta_2^*)'\, EX_i \right) \to 0$$

so the sum $\to 1$.

CASE 2) $(\theta_1-\theta_2^*)'\, EX_i < 0 \iff -\sqrt{n}\,(\theta_1-\theta_2^*)'\, EX_i \to \infty$ as $n \to \infty$

then

$$P\left(O_p(1) > \varepsilon - \sqrt{n}\,(\theta_1-\theta_2^*)'\, EX_i \right) \to 0$$

$$P\left(O_p(1) < -\varepsilon - \sqrt{n}\,(\theta_1-\theta_2^*)'\, EX_i \right) \to 1$$

so the sum $\to 1$.

Therefore, this family of random functions is NOT SE.

The previous examples were linear (in θ) functions. What happens when we have non-linear but differentiable functions?

<u>Example 4:</u> $H_n(\theta) = \frac{1}{n}\sum_{i=1}^{n} h(w_i,\theta)$, where $E \sup_{\theta\in\Theta} \left\| \frac{\partial h(w_i,\theta)}{\partial\theta} \right\| \leq K$.

We can linearize with mean value expansion and we're back to linear case!

$$H_n(\theta_1) - H_n(\theta_2) = \frac{1}{n}\sum_{i=1}^{n} \left(h(w_i,\theta_1) - h(w_i,\theta_2) \right)$$

$$= \frac{1}{n}\sum_{i=1}^{n} \frac{\partial h(w_i,\theta^*)}{\partial\theta'} (\theta_1 - \theta_2)$$

$$\Rightarrow |H_n(\theta_1) - H_n(\theta_2)| \leq \frac{1}{n}\sum_{i=1}^{n} \left\| \frac{\partial h(w_i,\theta^*)}{\partial\theta'} \right\| \|\theta_1 - \theta_2\|$$

$$\Rightarrow |H_n(\theta_1) - H_n(\theta_2)| \leq \underbrace{\frac{1}{n}\sum_{i=1}^{n} \sup_{\theta\in\Theta} \left\| \frac{\partial h(w_i,\theta)}{\partial\theta'} \right\|}_{= E \sup_{\theta\in\Theta} \left\| \frac{\partial h(w_i,\theta)}{\partial\theta'} \right\| + o_p(1) \leq K + o_p(1).} \|\theta_1 - \theta_2\|$$

$$\Rightarrow \sup_{\theta_2 \in B(\theta_1,\delta)} |H_n(\theta_1) - H_n(\theta_2)| \leq (K + o_p(1))\,\delta$$

Therefore

$$\lim_{n\to\infty} P\left(\sup_{\theta_1\in\Theta} \sup_{\theta_2\in B(\theta_1,\delta)} |H_n(\theta_1) - H_n(\theta_2)| > \varepsilon \right) \leq \lim_{n\to\infty} P\left(\underbrace{\delta K + o_p(1) > \varepsilon}_{\text{choose } \delta \text{ s.t.}} \right) \quad \varepsilon - \delta K > 0 \Rightarrow \delta < \frac{\varepsilon}{K}$$

$$= 0.$$

Example 5 : $H_n(\theta) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[h(w_i,\theta) - E h(w_i,\theta) \right]$, where

$$E \sup_{\theta \in \Theta} \left\| \frac{\partial h(w_i,\theta)}{\partial \theta} \right\| \leq K .$$

We can linearize with mean value expansion and we're back to linear case!

$$H_n(\theta_1) - H_n(\theta_2) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\left(h(w_i,\theta_1) - E h(w_i,\theta_1) \right) - \left(h(w_i,\theta_2) - E h(w_i,\theta_2) \right) \right]$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{\partial h(w_i,\theta^*)}{\partial \theta'} - \underbrace{\frac{\partial E h(w_i,\theta^*)}{\partial \theta'}}_{\text{interchange iff } E \sup_{\theta \in \Theta} h(w_i,\theta) < \infty} \right) (\theta_1 - \theta_2)$$

$$= \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[\frac{\partial h(w_i,\theta^*)}{\partial \theta'} - E \frac{\partial}{\partial \theta'} h(w_i,\theta^*) \right] (\theta_1 - \theta_2)$$

Then

$$\overline{\lim_{n\to\infty}} P\left(\sup_{\theta_1 \in \Theta} \sup_{\theta_2 \in B(\theta_1,\delta)} |H_n(\theta_1) - H_n(\theta_2)| > \varepsilon \right) \leq \overline{\lim_{n\to\infty}} P\left(O_p(1)\, \delta > \varepsilon \right)$$

choose δ s.t. $\varepsilon/\delta > M_\varepsilon$

$\Rightarrow$ $\delta < \varepsilon / M_\varepsilon$

$= 0 .$

Why is it useful? 1) It helps to achieve uniform law of large numbers (see old tutorial)

2) If $\hat{\beta_n} \xrightarrow{p} \beta_0$ we can replace $H_n(\hat{\beta_n})$ by $H_n(\beta_0)$ with $o_p(1)$ penalty.

Theorem.- Let $\hat{\beta_n} \xrightarrow{p} \beta_0$ and $\{H_n(\cdot), n \geq 1\}$ be SE.

Then

$$H_n(\hat{\beta_n}) = H_n(\beta_0) + o_p(1)$$

proof:

Write

$$\lim_{n\to\infty} P\left(|H_n(\hat{\beta_n}) - H_n(\beta_0)| > \varepsilon \right) \leq \lim_{n\to\infty} P\left(|H_n(\hat{\beta_n}) - H_n(\beta_0)| > \varepsilon,\ \hat{\beta_n} \in B(\beta_0, \delta) \right)$$

$$+$$

$$\lim_{n\to\infty} P\left(|H_n(\hat{\beta_n}) - H_n(\beta_0)| > \varepsilon,\ \hat{\beta_n} \notin B(\beta_0, \delta) \right)$$

$$\leq \overline{\lim_{n\to\infty}} P\left(\sup_{b_1 \in \Theta} \sup_{b_2 \in B(b_1, \delta)} |H_n(b_1) - H_n(b_2)| > \varepsilon \right)$$

$$+$$

$$\lim_{n\to\infty} P\left(\hat{\beta_n} \notin B(\beta_0, \delta) \right)$$

$< \varepsilon$, as desired.

When we encounter non-differentiable processes that are SE we can find a way to deal with an expectation, which is usually smoother.

For instance, in quantile regression we have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left(\tau - \underbrace{\mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\}}_{\text{non-differentiable!}}\right)X_i$$

non-differentiable! But the expectation

$$E[(\tau - \mathbb{1}\{Y_i < X_i'b\})X_i] \underset{LIE}{=} E[(\tau - F(X_i'b\,|\,X_i))X_i]$$

is differentiable!

From F.O.C:

$$o_p\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{n}\sum_{i=1}^{n}(\tau - \mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\})X_i$$

$$O_p(1) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}(\tau - \mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\})X_i \pm \sqrt{n}\,E[(\tau - F(X_i'\hat\beta_{\tau,n}\,|\,X_i))X_i]$$

$$O_p(1) = \underbrace{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{(\tau - \mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\})X_i - E[(\tau - F(X_i'\hat\beta_{\tau,n}\,|\,X_i))X_i]\right\}}_{H_n(\hat\beta_{\tau,n}) \text{ assume it's SE}} + \sqrt{n}\,E[(\tau - F(X_i'\hat\beta_{\tau,n}\,|\,X_i))X_i]$$

$$O_p(1) = \underbrace{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{(\tau - \mathbb{1}\{Y_i < X_i'\beta_0^\tau\})X_i - E[(\tau - F(X_i'\beta_0^\tau\,|\,X_i))X_i]\right\}}_{H_n(\beta_0^\tau)\text{, we can apply CLT!}} + \sqrt{n}\,\underbrace{E[(\tau - F(X_i'\hat\beta_{\tau,n}\,|\,X_i))X_i]}_{\substack{\text{This is a smooth function of } b,\\ \text{we can use mean value.}\\ \text{Call this function } m(b)}}$$

Then

$$O_p(1) = H_n(\beta_0^\tau) + \sqrt{n}\,m(\beta_0^\tau) + \sqrt{n}\,\underbrace{\frac{\partial m(\beta^*_{\tau,n})}{\partial\beta_\tau'}}_{\text{Assume continuous at } \beta_0^\tau.}(\hat\beta_{\tau,n} - \beta_0^\tau)$$

$$o_p(1) = \mathbb{H}_n(\beta_0^\tau) + \underbrace{\sqrt{n}\ m(\beta_0^\tau)}_{=0 \text{ under LIE}} + \sqrt{n}\,\frac{\partial m(\beta_0^\tau)}{\partial \beta_\tau'}\,(\hat{\beta}_{\tau,n} - \beta_0^\tau)$$

$$\Rightarrow \sqrt{n}\,(\hat{\beta}_{\tau,n} - \beta_0^\tau) = \left[\frac{\partial m(\beta_0^\tau)}{\partial \beta_\tau'}\right]^{-1} \underbrace{\mathbb{H}_n(\beta_0^\tau)}_{\xrightarrow{d} N(0,\ \Omega_0)} + o_p(1)$$

$$\xrightarrow{d} N\left(0,\ \left[\frac{\partial m(\beta_0^\tau)}{\partial \beta_\tau'}\right]^{-1} \Omega_0 \left[\frac{\partial m(\beta_0^\tau)}{\partial \beta_\tau}\right]^{-1}\right).$$

Showing such process $\mathbb{H}_n(\cdot)$ is SE is not so straightforward, but you can take a look at the Quantile Reg lecture.