Consider iid data $\{(Y_i, X_i')' : i=1, \dots, n\}$ and suppose the conditional distribution of Y_i given X_i is continous. Recall that $F(y \mid X_i) = P(Y_i < y \mid X_i)$.

We will assume that the conditional quantile of $Y_i \mid X_i$ is a parametric function of X_i:

$$q_\tau(X_i) = X_i'\beta_\tau \quad , \quad \beta_\tau \in \mathbb{R}^k$$

Then define

$$Q(b) = E\, \rho_\tau(Y_i - X_i'b)$$

$$Q_n(b) = \frac{1}{n}\sum_{i=1}^{n} \rho_\tau(Y_i - X_i'b) \xrightarrow[p]{} E\, \rho_\tau(Y_i - X_i'b)$$

where $\dfrac{\partial \rho_\tau(u)}{\partial u} = \tau - \mathbb{1}\{u<0\}$.

$$\int [-\tau + \mathbb{1}\{\delta - x < 0\}]\, \hat f_\delta(\delta)\, d\delta$$

$$\tau = \int_{-\infty}^{x} \hat f_\delta(\delta)\, d\delta$$

The FOC is:

$$\frac{\partial Q_n(b)}{\partial b} = \frac{1}{n}\sum_{i=1}^{n} \frac{\partial \rho_\tau(Y_i - X_i'b)}{\partial b} = -\frac{1}{n}\sum_{i=1}^{n} (\tau - \mathbb{1}\{Y_i - X_i'b < 0\})\, X_i$$

$$o_p(1/\sqrt{n}) = \frac{\partial Q_n(\hat\beta_{\tau,n})}{\partial \beta} = \underbrace{\frac{1}{n}\sum (\tau - \mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\})\, X_i}_{\text{Problem: non differentiable, our solution is to make it smooth using expectation for a fixed value } \hat\beta_{\tau,n}.}$$

Define

$$m(b) = E\left[\cancel{\frac{1}{n}}\,\cancel{\sum^{n}}\,(\tau - \mathbb{1}\{Y_i < X_i'b\})\, X_i\right]$$

$$\underset{\text{LIE}}{=} E\left[(\tau - F(X_i'b \mid X_i))\, X_i\right]$$

(notice that if evaluated at β_τ this is zero, i.e. $m(\beta_\tau) = 0$)

After adding and substracting

$$o_p\left(\frac{1}{\sqrt{n}}\right) = \frac{1}{n}\sum^{n}\left\{(\tau - \mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\})\, X_i - m(\hat\beta_{\tau,n})\right\} + m(\hat\beta_{\tau,n})$$

Multiplying $\sqrt{n}$

$$o_p(1) = \underbrace{\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\left\{(\tau - \mathbb{1}\{Y_i < X_i'\hat\beta_{\tau,n}\})\, X_i - m(\hat\beta_{\tau,n})\right\}}_{\nu_n(\hat\beta_{\tau,n}) \text{ and we assume this process is SE}} + \underbrace{\sqrt{n}\, m(\hat\beta_{\tau,n})}_{\text{now this is smooth! } (\sqrt{n}(\hat\beta_\tau - \beta_\tau))}$$

$$o_p(1) = v(\beta_\tau) + o_p(1) + \sqrt{n}\, m(\beta_{\tau,n}^{\hat{}})$$

Mean value expansion $\qquad \frac{\partial m(\beta_\tau)}{\partial \beta_\tau'} + o_p(1)$

$$o_p(1) = v(\beta_\tau) + o_p(1) + \sqrt{n}\,\underbrace{m(\beta_\tau)}_{=0} + \sqrt{n}\,\underbrace{\frac{\partial m(\beta_{\tau n}^{*})}{\partial \beta_\tau'}}\,(\hat{\beta}_{\tau,n} - \beta_\tau)$$

$F(X_i'\beta_\tau | X_i) = \tau$

We can use WLLN, the function is non random, only the argument is.

Rewrite

$$\sqrt{n}\,(\hat{\beta}_{\tau,n} - \beta_\tau) = \frac{\partial m(\beta_\tau)}{\partial \beta_\tau'}^{-1} \underbrace{v(\beta_\tau)}_{\xrightarrow{d} N(0,\Omega_0)} + o_p(1)$$

$$\xrightarrow{d} N\left(0, \frac{\partial m(\beta_\tau)}{\partial \beta_\tau'}^{-1} \Omega_0 \frac{\partial m(\beta_\tau)}{\partial \beta_\tau}^{-1\prime}\right).$$

(*) $v(\beta_\tau) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} [(\tau - 1\{y_i < x_i'\beta_0\}) x_i] \xrightarrow{d} N(0, E\, x_i x_i' (\tau - 1\{y_i < x_i'\beta_0\})^2)$

$N(0, E\, x_i x_i'\, \tau(1-\tau))$

$$\frac{\partial m(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta}\left[E\,(\tau - 1\{y_i \le x_i'\beta\}) X_i \right]$$

LIE $\qquad = \frac{\partial}{\partial \beta}\left[E\left\{ (\tau - F(x_i'\beta | x_i)) x_i \right\}\right]$

$\qquad = -E[f(x_i'\beta | X_i)\; x_i x_i']$ $\qquad$ which we assume to be full column rank k.

DCT

$\tau - F(x_i'\beta | x_i, \gamma_i) \le 1$

- Another way to see this model is

$$y_i = x_i'\beta_\tau + u_i$$
$$P(u_i \le 0 | x_i) = \tau$$

OLS

$y_i = x_i'\beta + u_i$

$E(u_i | X_i) = 0 \Rightarrow E[y_i | X_i] = X_i'\beta$

We have a conditional moment restriction given by

$$P(u_i \le 0 | x_i) = P(y_i \le x_i'\beta_\tau | X_i) = \tau$$

$$\Rightarrow \quad 0 = E[\tau - 1\{y_i - x_i'\beta_\tau \le 0\} | X_i]$$

This can be also rephrased as

$$E[(\tau - 1\{y_i - X_i'\beta_\tau\})\, g(x_i)] = 0$$

for any measurable function $g(\cdot)$.

So we are actually using $g(X_i) = x_i$, based on the check function approach.

- Introducing IVs

$$Y_i = X_i'\beta_\tau + u_i$$

$$P(u_i \leq 0 \mid z_i) = \tau \iff E[\mathbb{1}\{u_i \leq 0\} \mid z_i] = \tau$$

$$\iff E[\mathbb{1}\{u_i \leq 0\} - \tau \mid z_i] = 0$$

$$\iff E[(\tau - \mathbb{1}\{Y_i - X_i'\beta_\tau \leq 0\})\, g(z_i)] = 0$$

for any measurable $g(\cdot)$.
function

$$\sqrt{n}\, o_p\left(\frac{1}{\sqrt{n}}\right) = \sqrt{n}\,\frac{1}{n}\sum_{i=1}^{n}\left(\tau - \mathbb{1}\{Y_i \leq X_i'\hat{\beta}_{\tau,n}\}\right)h(z_i) \pm \sqrt{n}\, E\left(\tau - \mathbb{1}\{Y_i \leq X_i'\hat{\beta}_{\tau,n}\}\right)h(z_i)$$

$$o_p(1) = H_n(\hat{\beta}_{\tau,n}) + \sqrt{n}\, m(\hat{\beta}_{\tau,n})$$

We want to show $H_n(\hat{\beta}_{\tau,n}) = H_n(\beta_\tau) + o_p(1)$, i.e. for all $\varepsilon > 0$ $\lim_{n\to\infty} P(\|H_n(\hat{\beta}_{\tau,n}) - H_n(\beta_\tau)\| > \varepsilon) = 0$. Assume it is SE.

$$\lim_{n\to\infty} P(\|H_n(\hat{\beta}_{\tau,n}) - H_n(\beta_\tau)\| > \varepsilon) = P(\|H_n(\hat{\beta}_{\tau,n}) - H_n(\beta_\tau)\| > \varepsilon, \|\hat{\beta}_{\tau,n} - \beta_\tau\| < \delta)$$

$$+ P(\|H_n(\hat{\beta}_{\tau,n}) - H_n(\beta_\tau)\| > \varepsilon, \|\hat{\beta}_{\tau,n} - \beta_\tau\| \geq \delta)$$

$$\leq P(\|H_n(\hat{\beta}_{\tau,n}) - H_n(\beta_\tau)\| > \varepsilon, \|\hat{\beta}_{\tau,n} - \beta_\tau\| < \delta) + P(\underbrace{\|\hat{\beta}_{\tau,n} - \beta_\tau\|}_{o_p(1)} \geq \underbrace{\delta}_{>0})$$

$$\leq P(\underset{\text{sup sup}}{\|H_n(\hat{\beta}_{\tau,n}) - H_n(\beta_\tau)\|} > \varepsilon, \underset{\text{ignore this}}{\|\hat{\beta}_{\tau,n} - \beta_\tau\| < \delta}) + o(1)$$

$$\lim_{n\to\infty} \leq \underbrace{P\left(\sup_{b_1 \in B}\ \sup_{b_2 \in B(b_1,\delta)} \|H_n(b_1) - H_n(b_2)\| > \varepsilon\right)}_{<\varepsilon \text{ when taking limit by SE}} + \underbrace{o(1)}_{=0 \text{ when taking limit.}}$$

$$
\begin{aligned}
o_p(1) &= H_n(\hat\beta_{\tau,n}) + \sqrt{n}\; m(\hat\beta_{\tau,n}) \\
&= H_n(\beta_\tau) + o_p(1) + \sqrt{n}\; m(\hat\beta_{\tau,n}) \\
&= H_n(\beta_\tau) + o_p(1) + \sqrt{n}\left[\underbrace{m(\beta_\tau)}_{=0 \text{ by def}} + \frac{\partial m(\beta^*_{\tau n})}{\partial \beta_\tau} (\hat\beta_{\tau n} - \beta_{\tau n}) \right]
\end{aligned}
$$

where • $H_n(\beta_{\tau n}) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left[(\tau - \mathbb{1}\{y_i \le x_i'\beta_\tau\}) h(z_i) - \underbrace{E(\tau - \mathbb{1}\{y_i \le x_i'\beta_\tau\}) h(z_i)}_{=0 \text{ by LIE}} \right]$

$$
\begin{aligned}
&\xrightarrow[d]{} N(0, E(\tau - \mathbb{1}\{y_i \le x_i'\beta_\tau\})^2 h(z_i) h(z_i)') \\
&= N(0, E\{ E[(\tau - \mathbb{1}\{u_i \le 0\})^2 | z_i] h(z_i) h(z_i)'\}) \\
&= N(0, E\{ E[\underbrace{\tau^2 - 2\tau \mathbb{1}\{u_i \le 0\} + \mathbb{1}\{u_i \le 0\} | z_i}_{\tau^2 - 2\tau^2 + \tau = \tau - \tau^2 = \tau(1-\tau)}] h(z_i) h(z_i)'\}) \\
&= N(0, \tau(1-\tau) E h(z_i) h(z_i)')
\end{aligned}
$$

• $\frac{\partial m(\beta)}{\partial \beta} = \frac{\partial}{\partial \beta}\left[E(\tau - \mathbb{1}\{y_i \le x_i'\beta\}) h(z_i) \right]$

LIE $= \frac{\partial}{\partial \beta}\left[E\left\{ (\tau - F(x_i'\beta | x_i, z_i)) h(z_i) \right\} \right]$

DCT $= -E[f(x_i'\beta | z_i, x_i) h(z_i) x_i']$ which we assume to be full column rank k.

$\tau - F(x_i'\beta | x_i, z_i) \le 1$

Then

$$
\begin{aligned}
\sqrt{n}(\hat\beta_{\tau n} - \beta_{\tau n}) &= \left[-E[f(x_i'\beta | z_i, x_i) h(z_i) x_i']^{-1} + o_p(1) \right] \left[H_n(\beta_{\tau n}) + o_p(1) \right] \\
&= -E[f(x_i'\beta | z_i, x_i) h(z_i) x_i']^{-1} H_n(\beta_{\tau,n}) + o_p(1) \\
&\xrightarrow[d]{} N(0, V_\tau)
\end{aligned}
$$

where $V_\tau =$

$$\tau(1-\tau) \left(E[f(x_i'\beta_\tau | z_i, x_i) h(z_i) x_i']\right)^{-1} E h(z_i) h(z_i)' \left(E[f(x_i'\beta_\tau | z_i, x_i) x_i h(z_i)']\right)^{-1}$$

Rewrite the conditional moment restriction as

$$0 = \tau - P(y_i \le x_i'\beta_\tau \mid z_i)$$

$$= E[\underbrace{\tau - F(x_i'\beta_\tau \mid x_i, z_i)}_{m(x_i, z_i, \beta_\tau)} \mid z_i] \iff E[(\tau - F(x_i'\beta_\tau) \mid x_i, z_i)h(z_i)] = 0$$

for any $h(\cdot)$ measurable

Then

$$h^*(z_i) = \frac{1}{E[m^2(x_i, z_i, \beta_\tau) \mid z_i]} E\left[\frac{\partial m(x_i, z_i, \beta_\tau)}{\partial \beta} \mid z_i\right]$$

- $E[(\tau - F(x_i'\beta_\tau \mid x_i, z_i))^2 \mid z_i] = \tau(1-\tau)$
- $E\left[\frac{\partial m}{\partial \beta} \mid z_i\right] = -E[f(x_i'\beta_\tau \mid x_i, z_i)\, x_i \mid z_i]$

Then

$$h^*(z_i) = \frac{E[f(x_i'\beta_\tau \mid x_i, z_i)\, x_i \mid z_i]}{\tau(1-\tau)}$$

which implies that

$$V_{h^*} = \tau(1-\tau)\left\{E\left(E[f(x_i'\beta_\tau \mid x_i, z_i)\, x_i \mid z_i]\, E[f(x_i'\beta_\tau \mid x_i, z_i)\, x_i' \mid z_i]\right)\right\}^{-1}$$

(x) If $z_i = x_i$ as in standard quantile reg :

$$h^*(x_i) = \frac{f(x_i'\beta_\tau)\, x_i}{\tau(1-\tau)}$$ and the check function approach uses $h(x_i) = x_i$.